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PROBLEME DE MATEMATICĂ DE LICEU

ŞIRURI DE NUMERE REALE

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Introducere

Introducerea si studiul sirurilor de numere in scoala ocupa un loc esential in invatamantul matematic liceal.

Experienta didactica, dezvoltarea matematicii si efortul continuu de a mari eficienta actului pedagogic, au condus la forme consacrate si aproape general acceptate de a introduce sirurile de numere reale in matematica scolara.

Capitol de mare dificultate si finete capitolul “Siruri de numere. Criterii de convergenta.”- sta la baza introducerii notiunilor de limita unei functii, functii continue, functii derivabile, practic la baza intregii analize matematice.

Este cunoscuta importanta rezolvarii de exercitii si probleme pentru buna asimilare a oricarui domeniu al matematicii, deci cu atat mai mult cand este vorba de un domeniu cu un caracter mai abstract, asa cum este analiza matematica.

Mentionam ca intelegerea si utilizarea notiunii de limita (de limita a unui sir), este fundamentala pentru parcurgerea oricarui capitol din analiza matematica.

Capitolul “Siruri de numere” pune probleme reale la clasa , iar execitiile necesare intelegerii profunde si temeinice a acestei notiuni trebuiesc indelung cautate in diferite carti. Aceasta lucrare este in mare parte rodul unei astfel de cautari.

Capitolul I contine definitii, teoreme, corolare necesare abordarii unei probleme de siruri.

Capitolul II contine probleme diverse de diferite dificultati, necesare profesorului si utile elevului. Ele au fost selectate din culegeri, gazeta de matematica, unele fiind date ca subiecte la concursurile si olimpiadele scolare.

Capitolul III contine solutiile acestor probleme.

CAPITOLUL I
NOTIUNI TEORETICE.

Definitia 1
Fie M omultime fixata. Prin sir infinit de elemente ale lui M vom intelege o functie
f:$N_k \to M$, $N_k = \{n \in N / n > k\}$ unde k este un numar natural fixat.
Definitia 2
Fie $(a_n)_{n\geq 1}$ un sir de numere reale si $a \in R$. Se spune ca sirul $(a_n)_{n\geq 1}$ are limita "a" daca in orice vecinatate a punctului "a" se afla toti termenii sirului incepand de la un anumit rang. Se scrie atunci ca $\lim\limits_{n\to\infty} a_n = a$.
Definitia 3
Orice sir de numere reale avand o lim *ita finita se numeste convergent*.
Sirurile care nu au limita si cele care au limita $\pm\infty$ se numesc siruri divergente.
P_1. Un sir convergent are o singura limita.
1). Teorema de convergenta cu ε.
$a_n \to a$ daca si numai daca: pentru fiecare numar $\varepsilon > 0$ se poate gasi un rang
n_ε astfel incat pentru toti termenii de rang mai mare $(n > n_\varepsilon)$ sa avem : $|a_n - a| < \varepsilon$
2). Criteriul majorarii.
Daca $|a_n - a| \leq \alpha_n$ pentru orice n si daca $\alpha_n \to 0$ atunci $a_n \to a$.
3). Sirul modulelor.

Daca (a_n) este un sir convergent, atunci sirul $(|a_n|)$ este convergent si $\lim\limits_{n\to\infty}|a_n| = \left|\lim\limits_{n\to\infty} a_n\right|$.

Limita modulului este egala cu modulul limitei.
4). Trecerea la limita in inegalitati.
Daca (a_n) si (b_n) sunt doua siruri convergente si daca $a_n \leq b_n$ pentru orice
$n \in N$, atunci : $\lim\limits_{n\to\infty} a_n \leq \lim\limits_{n\to\infty} b_n$.
5). Teorema de convergenta a sirurilor monotone.
Orice sir monoton si marginit este convergent.
6). Teorema clestelui.

Daca $a_n \leq x_n \leq b_n$ pentru orice $n \in N$ si daca sirurile (a_n) si (b_n) sunt convergente si au aceiasi limita, atunci sirul (x_n) este convergent si are aceiasi limita ca si celelalte doua.
7). Criteriul general al lui Cauchy.
Un sir de numere (a_n) *este convergent* daca si numai daca pentru orice $\varepsilon > 0$ exista un numar
$N(\varepsilon)$ astfel incat oricare ar fi $n > N(\varepsilon)$ si orice p intreg $(p \geq 1)$ sa avem: $|a_{n+p} - a_n| < \varepsilon$.
8). Teorema

Daca $\lim\limits_{n\to\infty} a_n = a$, $a_n > 0$, atunci $\lim\limits_{n\to\infty} \dfrac{a_1 + a_2 + \cdots a_n}{n} = a$.

Daca $\lim\limits_{n\to\infty} a_n = a$, $a_n > 0$, atunci $\lim\limits_{n\to\infty} \sqrt[n]{a_1 \cdot a_2 \cdots a_n} = a$

9). Teorema

Daca $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = a$, $a_n > 0$, atunci $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = a$.

10). Teorema Stolz-Cezaro.

Fie a_n, b_n doua siruri de numere reale cu proprietatile :

1) $(b_n)_{n\geq 0}$ este strict monoton si nemarginit ($b_0 \in R^*$).

2) Exista $\lim\limits_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \lambda \in \overline{R}$, atunci exista si $\lim\limits_{n\to\infty} \frac{a_n}{b_n}$ *si avem* $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \lambda$.

11. Numarul e.

Sirul $a_n = \left(1+\frac{1}{n}\right)^n$, $n \geq 1$ este sir crescator si marginit $\Rightarrow a_n$ sir convergent.

$\lim\limits_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$. $2 \leq e \leq 3$.

CAPITOLUL AL II-LEA
APLICATII

A. Criterii de comparatie

1. Sa se calculeze limita sirului:

$$a_n = \frac{2^n}{n!} \quad , \quad \forall n \in N$$

2. Sa se calculeze limita sirului $\{a_n\}_{n\in N}$,

unde $a_n = \frac{1}{2^{2n}} \cdot C_{2n}^n$, $\forall n \in N$.

3. Sa se calculeze limita sirului:

$$a_n = \sum_{k=1}^{n} \frac{\cos k}{n^2+kn+k^2} \quad , \quad \forall n \in N^*.$$

4. Sa se calculeze limita sirului:

$$a_n = \frac{[x]+[2^2 x]+\cdots\cdots+[n^2 x]}{n^2} \quad , \quad \forall n \in N^*.$$

5. Sa se calculeze limita sirului:

$$a_n = \sum_{k=1}^{n} \left(\sqrt[3]{1+\frac{k^2}{n^3}}-1\right) \quad , \quad \forall n \in N^*.$$

6. Sa se calculeze limita sirului:

$$a_n = \sum_{k=1}^{n} \frac{\sin k}{n^2+k} \quad , \quad \forall n \in N^*.$$

7. Sa se calculeze limita sirului:

$$a_n = \frac{1^p + 2^p + 3^p + \cdots\cdots + n^p}{n^{p+1}}, \qquad p \in N.$$

8. Sa se calculeze: $\lim\limits_{n\to\infty} \frac{\ln n}{n}$

9. Sa se calculeze: $\lim\limits_{n\to\infty} \sqrt[n]{n}$

10. Sa se calculeze: $\lim\limits_{n\to\infty} \frac{n}{\sqrt[n]{n!}}$

11. Sa se calculeze: $\lim\limits_{n\to\infty} \left(\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!}\right)$

12. Sa se calculeze: $\lim\limits_{n\to\infty} \frac{n^n}{n!}$

13. Sa se calculeze limita sirului:

$$a_n = \sqrt[n]{\frac{3^{3n}(n!)^3}{(3n)!}}, \qquad \forall n \in N^*.$$

14. Sa se calculeze limita sirului:

$$a_n = \frac{2^3 - 1}{2^3 + 1} \cdot \frac{3^3 - 1}{3^3 + 1} \cdot \frac{4^3 - 1}{4^3 + 1} \cdots \frac{n^3 - 1}{n^3 + 1}$$

15. Sa se calculeze limita sirului:

$$a_n = \sum_{k=2}^{n} \log_{\frac{1}{3}} \left(1 - \frac{2}{k(k+1)}\right)$$

16. Sa se calculeze limita sirului b_n:

$$a_n = n - \sum_{k=1}^{n} \frac{k^2 + 3k + 1}{k^2 + 3k + 2}, \qquad b_n = \prod_{i=1}^{n} a_i \quad .$$

17. Sa se calculeze termenul general si apoi limita sirului a_n, unde

$$a_n = \frac{1}{3^2} + \frac{2}{15^2} + \frac{3}{35^2} + \cdots + \frac{n}{[(2n-1)(2n+1)]^2}$$

18. Sa se calculeze limitele sirurilor $(a_n)_{n\in N}$ si $(b_n)_{n\in N}$, cu termenii generali :

$$a_n = \sum_{k=1}^{n} \frac{k^2 + k - 1}{(k+2)!} \quad \text{si} \quad b_n = \frac{k^3 + k^2 - 2k - 1}{[(k+1)!]^2}, \qquad \forall n \in N$$

19. Sa se calculeze limita sirului cu termenul general:

$$a_n = \frac{2}{n\left[\left(1+\frac{1}{n}\right)^k - 1\right]}$$

B. Relatii de recurenta.

20. Sa se calculeze limita sirului:

$$x_{n+1} = \frac{x_n}{1+x_n}, x_0 = 1.$$

21. Sa se calculeze limita sirului:

$$x_{n+1} = \frac{x_n^2}{1+x_n}, x_0 = 1.$$

22. Sa se calculeze limita sirului:

$$x_{n+1} = \frac{ax_n}{1+bx_n}, x_0 = 1. \ n \geq 0, \ a>0, \ b \geq 0.$$

23. Sa se calculeze limita sirului:

$$x_{n+1} = \frac{1}{2}\left(x_n^2 + b\right), \ x_0 = 0, \ n \geq 0, \ b \in [0,1].$$

24. Sa se calculeze limita sirului:

$$x_n = \sqrt{\alpha + x_{n-1}}, \ x_1 = \sqrt{\alpha}, \ \alpha > 0, \ n \geq 2.$$

25. Sa se calculeze limita sirului:

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{1}{x_n}\right), \ x_1 = a, \ a>1.$$

26. Sa se studieze convergenta sirului:

$$x_{n+1} = 1 + \frac{x_n^2}{1+x_n}, x_0 = 1.$$

27. Sa se calculeze limita sirului:

$$x_{n+1} = x_n^2 - 2x_n + 2 \quad, x_0 = a.$$

28. Sa se calculeze limita sirului:

$$x_n = \frac{2ax_{n-1}}{x_{n-1} + a}, \ a, x_0 \in R \ .$$

29. Sa se studieze natura sirului $(a_n)_{n \in N,}$ definit astfel:

$$a_{n+1} = \sqrt{1 - a_n}, \ a_1 = a, \ \forall n \geq 1, \ a \in (0,1)$$

30. Se da sirul cu termenul general:
    a. $x_n = \sqrt{a + \sqrt{a + \cdots \sqrt{a}}}$, a>0, (n radicali).
    b. Sa se arate ca sirul este convergent si sa i-se calculeze limita.

31. Fie sirul $(x_n)_{n \in N}$, cu $x_1 = 1$, $x_n = 1 + \frac{n+1}{2n} \cdot \left(x_{n-1}\right)$, $\forall n \in N$, n>1.

    Sa se studieze natura sirului.

32. Fie $a_1 > 0$; $a_{n+1} = \frac{a_n}{1 + na_n^2}$

a. Sa se calculeze $\lim_{n\to\infty} a_n$

b. Sa se calculeze $\lim_{n\to\infty} na_n$

33. Daca $x_1 \in (0, \pi)$, $x_{n+1} = 2\cdot arctg\,\dfrac{2}{\sin x_n}$

a. Sa se calculeze $\lim_{n\to\infty} x_n$

b. Sa se calculeze $\lim_{n\to\infty} \sqrt{n}\sin x_n$

34. Se considera sirul de functii $(f_n)_{n\geq 1}$

$f_n = f\circ f\circ f\circ f\circ\cdots\circ f$ unde $f:R\to R$ si $f(x)=\dfrac{2x}{1+x^2}$

Sa se arate ca $\exists g : R\to R$ a.i. $\forall x\in R$ $\lim_{n\to\infty} f_n(x) = g(x)$.

35. Fie sirurile : $a_n, a_{n+1} = \dfrac{b_n}{a_n}$ si b_n, $b_{n+1} = \dfrac{b_n - 1}{a_n - 1}$ cu $b_1 > a_1 > 1$.

Sa se determine valorile lui a_1 si b_1, pentru care sirurile sant converg$\in R\in$ente.

36. Se considera sirul de numere reale, $(a_n)_{n\in N,}$ definit astfel: $a_{n+1}\cdot a_n + a_{n+1} = a_n - 1,\ \forall n\in N$,

Sa se stabileasca daca sirul are limita sau nu. Daca $a_{1976} = 40$, care este sirul?.

37. Sa se arate ca un sir periodic nu este convergent.

38. Se da sirul $a_n = \sum_{i=1}^{k}(-1)^{i+1}a_{n-i}$.

a. Sa se arate ca sirul este periodic $\forall k\geq 2$, $k\in N$.

b. Sa se gaseasca expresia termenului general al sirului $(a_n)_{n\in N}$, pentru k=2.

C. Siruri diverse

39.

Fie sirul $(\mathrm{I_n})_{n\geq 1}$, $\mathrm{I_n} = \int_n^{n+1}\dfrac{2x-1}{x}dx.$

a). Aratati ca sirul $(\mathrm{I_n})_{n\geq 1}$ este crescator.

b). Aratati ca sirul $(\mathrm{I_n})_{n\geq 1}$ este marginit.

c). Calculati $\lim_{\mathrm{n}\to\infty} n(2-I_n)$.

40. Fie sirul cu termenul general $u_n = \left(1+\dfrac{v_n}{n}\right)^n$, unde $v_n = \sum_{k=1}^{n}\dfrac{1}{(2k-1)(2k+1)}$,

a. Sa se calculeze limita sirului u_n

b. Sa se calculeze : $\lim_{n\to\infty}\dfrac{u_{n+1}}{u_n}$

41. Fie sirurile: $a_n = \left(\dfrac{n^2 - nk + 1}{n^2+1}\right)^{\frac{n^2+n+1}{n+1}}$, $b_n = \lim_{n\to\infty} a_n$.

Sa se calculeze limita sirului $(c_n)_{n\in n}$, unde $c_n = \sum_{k=1}^{n} b_k$.

42. Se considera sirul cu termenul general:

$$a_n = \frac{(n+3)^2+3}{n(n+1)(n+2)} \cdot \frac{1}{2^{n+1}}$$

Se considera apoi sirurile:

$$b_n = \sum_{k=1}^{n} a_k \text{ si } c_n = b_n^{2^n}.$$

Sa se calculeze limita celor trei siruri.

43. Fie sirul $(a_n)_{n\in N}$ convergent, definit astfel: $a_1 = a$, $a_{n+1} = \dfrac{a_n}{\sqrt{1+a_n^2}}$.

    a. Sa se calculeze limita sirului $(a_n)_{n\in N}$

    b. Sa se studieze natura sirului definit astfel: $b_n = \left(n \cdot a_{n+1}^2\right)^{-\alpha n-\beta}$, $\alpha, \beta \in R$.

44. Sa se calculeze limita sirului:

$$a_n = \frac{3^{3n}}{\left(3+\frac{1}{n}\right)^{3n}}$$

45. Sa se calculeze termenul general al sirului:

$$a_n = \cfrac{1+\cfrac{3+\cfrac{5+\cfrac{7+\cdots^{\vdots + \frac{2n-1}{2n}}}{8}}{6}}{4}}{2}, \ \forall n \in N$$

46. Sa se gaseasca termenul general al sirului dat de relatia:

$$\frac{1}{\sqrt[n+1]{a_{n+1}}-1} - \frac{1}{\sqrt[n]{a_n}-1} = 1, \ a_1 = 2.$$

47. Sa se demonstreze ca sirul:

$$a_n = 1 + \frac{1}{2^2} + \cdots + \frac{1}{n^2} \quad \text{este convergent.}$$

48. Sa se arate ca sirul $(x_n)_{n\in N}$, cu $x_n = \sum_{k=1}^{n} \dfrac{\cos a_k}{k(k+1)}$ este convergent, $(a_k)_{k\in N}$ fiind un sir de numere reale.

49. Fie sirul $(x_n)_{n\in N}$ cu $x_n = \dfrac{\cos 1!}{1\cdot 2} + \dfrac{\cos 2!}{2\cdot 3} + \cdots + \dfrac{\cos n!}{n(n+1)}$.

    Sa se arate ca sirul x_n este convergent.

D. Calculul limitelor unui sir cu ajutorul integralei definite.

50. Sa se calculeze limita sirului cu termenul general: $a_n = \dfrac{3}{n} \sum_{k=1}^{n} \sqrt{\dfrac{n}{n+3(k-1)}}$.

51. Sa se calculeze limita sirului cu termenul general: $a_n = \dfrac{1}{n} \sum_{k=1}^{n} \sqrt{1+\dfrac{k}{n}}$.

52. Sa se calculeze limita sirului cu termenul general: $a_n = \dfrac{\pi}{n} \sum_{k=1}^{n} \sin \dfrac{(k-1)\pi}{n}$

53. Sa se calculeze limita sirului cu termenul general: $a_n = n\sum_{k=1}^{n}\frac{1}{(n+k)^2}$.

54. Sa se calculeze limitele sirurilor cu termenii generali:

$$a_n = \frac{1}{n}\sum_{k=1}^{n} e^{\frac{k}{n}}\cos^2\frac{k}{n};\ b_n = \sum_{k=1}^{n} e^{\frac{k}{n}}\sin^2\frac{k}{n}.$$

55. Sa se arate ca: $\lim_{n\to\infty}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2n-1}-\frac{1}{2n}\right)=\ln 2$

CAPITOLUL AL III-LEA
REZOLVARI

1. $0<a_n = \frac{2}{1}\cdot\frac{2}{2}\cdot\frac{2}{3}\cdot\cdots\cdot\frac{2}{n} < \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{2}{3}\cdots\ \frac{2}{3} = 2\cdot\left(\frac{2}{3}\right)^{n-2}$, dar $\lim_{n\to\infty} 2\cdot\left(\frac{2}{3}\right)^{n-2}=0 \Rightarrow \lim_{n\to\infty} a_n = 0$.

2. $0<a_n = \frac{1}{2^{2n}}\cdot C_{2n}^{n} = \frac{1}{2^{2n}}\cdot\frac{(2n)!}{(n!)^2} = \frac{1\cdot 2\cdot 3\cdot 4\cdots(2n-1)\cdot 2n}{[2\cdot 4\cdot 6\cdots 2n]^2} = \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2\cdot 4\cdot 6\cdots 2n} < \frac{1}{\sqrt{2n+1}}$;

$$\lim_{n\to\infty}\frac{1}{\sqrt{2n+1}} = 0 \Rightarrow \lim_{n\to\infty} a_n = 0$$

3. $|a_n| = \left|\sum_{k=1}^{n}\frac{\cos k}{n^2+kn+k^2}\right| \le \sum_{k=1}^{n}\left|\frac{\cos k}{n^2+kn+k^2}\right| \le \sum_{k=1}^{n}\frac{1}{n^2+n+1} = \frac{n}{n^2+n+1} \to 0$

Conform criteriului majorarii, $\lim_{n\to\infty} a_n = 0$.

4. $x-1<[x]\le x$ deci: $n^2x-1<[n^2x]\le n^2x$.

n=1 $\quad x-1<[x]\le x$

n=2 $\quad 2^2x-1<[2^2x]\le 2^2x$.

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n=n $\quad n^2x-1<[n^2x]\le n^2x$.

$$x(1^2+2^2+\cdots+n^2)-n < [x]+[2^2x]+\cdots[n^2x] \le x(1^2+2^2+\cdots+n^2).$$

$$\frac{x(1^2+2^2+\cdots+n^2)-n}{n^3} < \frac{[x]+[2^2x]+\cdots[n^2x]}{n^3} \le \frac{x[1^2+2^2+\cdots n^2]}{n^3}.$$

$$\lim_{n\to\infty}\frac{x(1^2+2^2+\cdots+n^2)-n}{n^3} < \lim_{n\to\infty}\frac{xn(n+1)(2n+1)-6n}{6n^3} = \frac{2x}{6} = \frac{x}{3}$$

$$\lim_{n\to\infty}\frac{x(1^2+2^2+\cdots+n^2)}{n^3} < \lim_{n\to\infty}\frac{xn(n+1)(2n+1)}{6n^3} = \frac{2x}{6} = \frac{x}{3}$$

$$\lim_{n\to\infty}\frac{[x]+[2^2x]+\cdots+[n^2x]}{n^3} = \frac{x}{3}$$

5. $a_n=\sum_{k=1}^{n}\left(\sqrt[3]{1+\frac{k^2}{n^3}}-1\right)$

$$\lim_{n\to\infty}\frac{(n+1)(2n+1)}{6n^2\left(\sqrt[3]{1+\frac{1}{n}}^2+\sqrt[3]{1+\frac{1}{n}}+1\right)}=\frac{2}{18}=\frac{1}{9}$$

$$a_n>\frac{(n+1)(2n+1)}{6n^2\left(\sqrt[3]{1+\frac{1}{n}}^2+\sqrt[3]{1+\frac{1}{n}}+1\right)}$$

$$=\frac{1}{n^3}\cdot\sum_{k=1}^{n}\frac{k^2}{\sqrt[3]{1+\frac{k^2}{n^3}}^2+\sqrt[3]{1+\frac{k^2}{n^3}}+1}<\frac{\sum_{k=1}^{n}k^2}{n^3\left(\sqrt[3]{1+\frac{1}{n^3}}^2+\sqrt[3]{1+\frac{1}{n^3}}+1\right)}=\frac{(n+1)(2n+1)}{6n^2\left(\sqrt[3]{1+\frac{1}{n^3}}^2+\sqrt[3]{1+\frac{1}{n^3}}+1\right)}$$

Rezulta ca $\lim_{n\to\infty}a_n=\frac{1}{9}$.

6. $a_n=\sum_{k=1}^{n}\frac{\sin k}{n^2+k}<\sum_{k=1}^{n}\frac{1}{n^2+k}<\frac{n}{n^2+1}$

$a_n=\sum_{k=1}^{n}\frac{\sin k}{n^2+k}>-\sum_{k=1}^{n}\frac{1}{n^2+k}>-\frac{n}{n^2+1}$

Trecand la limita in inegalitatea de mai jos, se obtine:

$$-\frac{n}{n^2+1}<a_n<\frac{n}{n^2+1}$$

$$\lim_{n\to\infty}a_n=0$$

7. $a_n=\frac{1^p+2^p+3^p+\cdots\cdots+n^p}{n^{p+1}}$

Se noteaza $n^{p+1}=b_n$,

b_n -este sir crescator, divergent.

$$\lim_{n\to\infty}\frac{c_{n+1}-c_n}{b_{n+1}-b_n}=\lim_{n\to\infty}\frac{(n+1)^p}{(n+1)^{p+1}-n^{p+1}}=\lim_{n\to\infty}\frac{(n+1)^p}{C_{p+1}^1\cdot n^p+C_{p+1}^2\cdot n^{p-1}+\cdots+1}=\frac{1}{C_{p+1}^1}=\frac{1}{p+1},$$

Conform teoremei lui Stolz-Cezaro $\Rightarrow \lim_{n\to\infty}a_n=\frac{1}{p+1}$.

8. Fie $a_n=\ln n, b_n=n$ crescator si nemarginit.

$$\lim_{n\to\infty}\frac{a_{n+1}-a_n}{b_{n+1}-b_n}=\lim_{n\to\infty}\frac{\ln\frac{n+1}{n}}{n+1-n}=\ln 1=0$$

Din teorema lui Stolz-Cezaro $\Rightarrow \lim_{n\to\infty}\frac{\ln n}{n}=0.$

9. Fie a_n =n>0

$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{n+1}{n}=1\Rightarrow$ (din criteriul radicalului): $\lim_{n\to\infty}\sqrt[n]{a_n}=1.$ deci $\lim_{n\to\infty}\sqrt[n]{n}=1.$

10. $\lim\limits_{n\to\infty}\frac{n}{\sqrt[n]{n!}}=\lim\limits_{n\to\infty}\sqrt[n]{\frac{n^n}{n!}}$. Notam $a_n=\frac{n^n}{n!}$;

$\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim\limits_{n\to\infty}\frac{(n+1)^{n+1}}{(n+1)!}\cdot\frac{n!}{n^n}=\lim\limits_{n\to\infty}\frac{(n+1)^n}{n^n}=\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e\Rightarrow$ (din criteriul radicalului) :

$\lim\limits_{n\to\infty}\frac{n}{\sqrt[n]{n!}}=e.$

11. Fie $b_n=\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!}$; $a_n=\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}$

$\lim\limits_{n\to\infty}a_n=\lim\limits_{n\to\infty}\frac{\sqrt[n+1]{(n+1)!}}{n+1}\cdot\frac{n}{\sqrt[n]{n!}}\cdot\frac{n+1}{n}=\frac{1}{e}\cdot e\cdot 1=1$

$\lim\limits_{n\to\infty}(a_n)^n=\lim\limits_{n\to\infty}\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)^n=\lim\limits_{n\to\infty}\sqrt[n+1]{\frac{[(n+1)!]^n}{(n!)^{n+1}}}=\lim\limits_{n\to\infty}\sqrt[n+1]{\frac{(n+1)^n}{n!}}=\lim\limits_{n\to\infty}\sqrt[n]{\frac{n^{n-1}}{(n-1)!}}$;

Notam cu $c_n=\frac{n^{n-1}}{(n-1)!}$; $\lim\limits_{n\to\infty}\frac{c_{n+1}}{c_n}=\lim\limits_{n\to\infty}\frac{(n+1)^n}{n!}\cdot\frac{(n-1)!}{n^{n-1}}=\lim\limits_{n\to\infty}\frac{(n+1)^n}{n^n}=\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e$

Din criteriul raportului$\Rightarrow\lim\limits_{n\to\infty}c_n=e$ deci $\lim\limits_{n\to\infty}(a_n)^n=e$.

Dar $(a_n)^n=(1+a_n-1)^{\frac{1}{a_n-1}\cdot\frac{n}{\sqrt[n]{n!}}\cdot b_n}$; deoarece $\lim\limits_{n\to\infty}(a_n)^n=e$, $\lim\limits_{n\to\infty}(1+(a_n-1))^{\frac{1}{a_n-1}}=e$ si

$\lim\limits_{n\to\infty}\frac{n}{\sqrt[n]{n!}}=e,$

Rezulta ca b_n este convergent si trecand la limita in relatia anterioara obtinem: $e=e^{e\lim\limits_{n\to\infty}b_n}$,

deci $\lim\limits_{n\to\infty}b_n=\frac{1}{e}$.

12 $\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim\limits_{n\to\infty}\frac{(n+1)^{n+1}}{(n+1)!}\cdot\frac{n!}{n^n}=\lim\limits_{n\to\infty}\left(\frac{n+1}{n}\right)^n=e>1\Rightarrow$ (din criteriul raportului) ca $\lim\limits_{n\to\infty}a_n=\infty$

13 Fie

$a_n=\frac{3^{3n}(n!)^3}{(3n)!}$; $\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim\limits_{n\to\infty}\frac{3^{3n+3}[(n+1)!]^3}{(3n+3)!}\cdot\frac{(3n)!}{3^{3n}(n!)^3}=\lim\limits_{n\to\infty}\frac{3^3(n+1)^3}{(3n+1)(3n+2)(3n+3)}=\frac{27}{27}=1$

$\Rightarrow\lim\limits_{n\to\infty}\sqrt[n]{a_n}=1.$

14.

$a_n=\frac{2^3-1}{2^3+1}\cdot\frac{3^3-1}{3^3+1}\cdot\frac{4^3-1}{4^3+1}\cdot\ldots\cdot\frac{n^3-1}{n^3+1}=$

$\prod_{k=2}^{n}\frac{k^3-1}{k^3+1}=\prod_{k=2}^{n}\frac{(k-1)(k^2+k+1)}{(k+1)(k^2-k+1)}=\frac{1\cdot 7}{3\cdot 3}\cdot\frac{2\cdot 13}{5\cdot 13}\cdot\ldots\cdot\frac{(n-2)(n^2-n+1)}{n(n^2-3n+3)}\cdot\frac{(n-1)(n^2+n+1)}{(n+2)(n^2-n+1)}=$

$=\frac{2}{3}\cdot\frac{(n^2+n+1)}{n(n+1)}$

$\lim\limits_{n\to\infty}a_n=\frac{2}{3}$

15.

$$a_n=\sum_{k=2}^{n}\log_{\frac{1}{3}}\left(1-\frac{2}{k(k+1)}\right)=\sum_{k=2}^{n}\log_{\frac{1}{3}}\left(\frac{k^2+k-2}{k(k+1)}\right)=\sum_{k=2}^{n}\log_{\frac{1}{3}}\left(\frac{(k-1)(k+2)}{k(k+1)}\right)=\log_{\frac{1}{3}}\prod_{k=2}^{n}\frac{(k-1)(k+2)}{k(k+1)}=$$

$$\log_{\frac{1}{3}}\frac{1\cdot 4}{2\cdot 3}\cdot\frac{2\cdot 5}{3\cdot 4}\cdots\frac{(k-2)(k+1)}{k(k+1)}\cdot\frac{(k-1)(k+2)}{k(k+1)}=\log_{\frac{1}{3}}\frac{k+2}{3k};\Rightarrow\lim_{n\to\infty}a_n=1.$$

16.

$$a_n=n-\sum_{k=1}^{n}\frac{k^2+3k+1}{k^2+3k+2}=\sum_{k=1}^{n}\left(1-\frac{k^2+3k+1}{k^2+3k+2}\right)=\sum_{k=2}^{n}\frac{1}{(k+1)(k+2)}=\sum_{k=2}^{n}\left(\frac{1}{k+1}-\frac{1}{k+2}\right)=\frac{1}{2}-\frac{1}{n+2};\Rightarrow$$

$$\lim_{n\to\infty}a_n=\frac{1}{2}$$

17.

$$a_n=\frac{1}{3^2}+\frac{2}{15^2}+\frac{3}{35^2}+\cdots+\frac{n}{[(2n-1)(2n+1)]^2}=$$

$$\sum_{k=1}^{n}\frac{k}{[(2k-1)(2k+1)]^2}=\frac{1}{8}\sum_{k=1}^{n}\left[\frac{1}{(2k-1)^2}-\frac{1}{(2k+1)^2}\right]=\frac{n^2+n}{2(2n+1)^2}\Rightarrow\lim_{n\to\infty}a_n=\lim_{n\to\infty}\frac{n^2+n}{2(2n+1)^2}=\frac{1}{8}$$

18. $$a_n=\sum_{k=1}^{n}\frac{k^2+k-1}{(k+2)!}=\sum_{k=1}^{n}\left[\frac{k}{(k+1)!}-\frac{k+1}{(k+2)!}\right]=\frac{1}{2}-\frac{n+1}{(n+2)!}\Rightarrow\lim_{n\to\infty}a_n=\frac{1}{2}$$

$$b_n=\sum_{k=1}^{n}\frac{k^3+k^2-2k-1}{[(k+1)!]^2}=\sum_{k=1}^{n}\left\{\frac{k-1}{(k!)^2}-\frac{k}{[(k+1)!]^2}\right\}=-\frac{n}{[(n+1)!]^2}\Rightarrow\lim_{n\to\infty}b_n=0$$

19. $$a_n=\frac{2}{n\left[\left(1+\frac{1}{n}\right)^k-1\right]}=\frac{2}{\left[\left(1+\frac{1}{n}\right)^{k-1}+\left(1+\frac{1}{n}\right)^{k-2}+\cdots+1\right]}$$

$$\lim_{n\to\infty}a_n=2\lim_{n\to\infty}\frac{1}{\left[\left(1+\frac{1}{n}\right)^{k-1}+\left(1+\frac{1}{n}\right)^{k-2}+\cdots+1\right]}=\frac{2}{k}$$

20. $x_{n+1}-x_n=\dfrac{x_n}{1+x_n}-x_n=\dfrac{-x_n^2}{1+x_n}<0$ deoarece $x_n>0$ (se demonstreaza prin inductie completa)

$x_{n+1}<x_n\Rightarrow x_n$ sir monoton descrescato

$0<x_n$, x_n marginit inferior $\Rightarrow x_n$ *convergent*

Fie $\lim_{n\to\infty}x_n=x.$ Trecand la limita in egalitatea:

$$x_{n+1}=\frac{x_n}{1+x_n}\Rightarrow x=\frac{x}{x+1}\Rightarrow x^2+x=x\Rightarrow x=0\Rightarrow\lim_{n\to\infty}x_n=0$$

21. $x_{n+1}-x_n=\dfrac{x_n^2}{1+x_n}-x_n=\dfrac{-x_n}{1+x_n}<0$ deoarece $x_n>0$ (se demonstreaza prin inductie completa)

$x_{n+1}<x_n\Rightarrow x_n$ sir monoton descrescator.

$0<x_n$, x_n marginit inferior $\Rightarrow x_n$ *convergent*. Fie $\lim_{n\to\infty}x_n=x.$ Trecand la limita in egalitatea:

$$x_{n+1}=\frac{x_n^2}{1+x_n}\Rightarrow x=\frac{x^2}{x+1}\Rightarrow x^2+x=x^2\Rightarrow x=0\Rightarrow\lim_{n\to\infty}x_n=0$$

22. 1. b=0, a $\in (0,1)$

$x_{n+1} = ax_n$; $\frac{x_{n+1}}{x_n} = a < 1 \Rightarrow x_n$ monoton descrescator.

0<x_n; x_n marginit inferior $\Rightarrow x_n$ convergent. Fie $\lim\limits_{n\to\infty} x_n = x$. Trecand la limita in inegalitatea

$x_{n+1} = ax_n \Rightarrow$ x=ax $\Rightarrow$ x(a-1)=0 $\Rightarrow \lim\limits_{n\to\infty} x_n = 0$

2. b=0, a=1

$x_{n+1} = x_n = 1$ sir constant $\Rightarrow \lim\limits_{n\to\infty} x_n = 1$

3. b=0, a>1 $\Rightarrow x_n = a^n$ crescator si nemarginit $\Rightarrow \lim\limits_{n\to\infty} x_n = \infty$

4. b $\neq 0$

$$x_{n+1} - x_n = \frac{ax_n}{1+bx_n} - \frac{ax_{n-1}}{1+bx_{n-1}} = \frac{a(x_n - x_{n-1})}{(1+bx_n)(1+bx_{n-1})}$$

4′. a>1+b

$x_{n+1} - x_n > 0 \Rightarrow$ sirul x_n este monoton crescator (se demonstreaza prin inductie completa)

$x_{n+1} = \frac{ax_n}{1+bx_n} < \frac{ax_n}{bx_n} = \frac{a}{b} \Rightarrow x_n$ marginit superior. Deci x_n este convergent.

Fie $\lim\limits_{n\to\infty} x_n = x$. Trecand la limita in egalitatea :

$$x_{n+1} = \frac{ax_n}{1+bx_n} \Rightarrow x = \frac{ax}{1+bx} \Rightarrow bx^2 + x - ax - 0 \Rightarrow x(bx+1-a) = 0 \Rightarrow x = 0, x = \frac{a-1}{b} \Rightarrow$$

$$\lim\limits_{n\to\infty} x_n = \frac{a-1}{b}$$

4''. a<1+b

$x_{n+1} - x_n < 0 \Rightarrow$ sirul x_n este monoton descrescator(se demonstreza prin inductie completa).

0< $x_n \Rightarrow$ sirul x_n este marginit inferior $\Rightarrow$ x_n convergent. Fie $\lim\limits_{n\to\infty} x_n = x$.

Trecand la limita in egalitatea:

$$x_{n+1} = \frac{ax_n}{1+bx_n} \Rightarrow x = \frac{ax}{1+bx} \Rightarrow bx^2 + x - ax - 0 \Rightarrow x(bx+1-a) = 0 \Rightarrow x = 0, x = \frac{a-1}{b}$$

P(n): $x_n > \frac{a-1}{b}$

P(!): $x_1 > \frac{a-1}{b}$; $\frac{a}{1+b} > \frac{a-1}{b} \Rightarrow b+1 > 0 \Rightarrow P(1)$ adevarata.

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P(k): $x_k > \frac{a-1}{b}$

P(k+1): $x_{k+1} > \frac{a-1}{b}$;

$\frac{ax_n}{1+bx_n} > \frac{a-1}{b} \Rightarrow x_n > \frac{a-1}{b} \Rightarrow P(k) \Rightarrow P(k+1) \Rightarrow P(n)$ adevarata pentru $\forall$n $\in$ N $\Rightarrow \lim\limits_{n\to\infty} x_n = \frac{a-1}{b}$

23. $x_{n+1} - x_n = \frac{1}{2}(x_n^2 - b) - \frac{1}{2}(x_{n-1}^2 - b) = \frac{1}{2}(x_n^2 - x_{n-1}^2) = \frac{1}{2}(x_n - x_{n-1})(x_n + x_{n-1}) \Rightarrow x_{n+1} > x_n$ (se demonstreza prin inductie completa) $\Rightarrow$ x_n este monoton crescator.

x_n <b(se demonstreaza prin inductie completa) sirul x_n este marginit superior) $\Rightarrow$ x_n este convergent. Fie $\lim\limits_{n\to\infty} x_n = x.$ Trecand la limita in egalitatea $x_{n+1} = \frac{1}{2}\left(x_n^2 + b\right) \Rightarrow x = \frac{1}{2}\left(x^2 + b\right)$

$\Rightarrow x^2 - 2x + b = 0 \Rightarrow x_{1,2} = 1 \pm \sqrt{1-b} \Rightarrow \lim\limits_{n\to\infty} x_n = 1+\sqrt{1-b}$.

24.

$$x_{n+1} - x_n = \sqrt{\alpha + x_n} - \sqrt{\alpha + x_{n-1}} = \frac{x_n - x_{n-1}}{\sqrt{\alpha + x_n} + \sqrt{\alpha + x_{n-1}}} > 0$$

$x_n - x_{n-1} > 0$ (se demonstreaza prin inductie completa) $\Rightarrow x_n$ este monoton crescator.

$x_n = \sqrt{\alpha + x_{n-1}}$; $x_n^2 = \alpha + x_{n-1}$; $x_n^2 - x_{n-1} - \alpha = 0 \Rightarrow x_n^2 - x_n - \alpha < 0$

$(x_n)_{1,2} = 1 \pm \sqrt{1+4\alpha} \Rightarrow (x_n) \in (1-\sqrt{1+4\alpha}, 1+\sqrt{1+4\alpha}) \Rightarrow x_n$ marginit $\Rightarrow$ x_n convergent.

Fie $\lim\limits_{n\to\infty} x_n = x.$ Trecand la limita in egalitatea:

$x_n = \sqrt{\alpha + x_{n-1}} \Rightarrow x = \sqrt{\alpha + x} \Rightarrow x^2 = \alpha + x \Rightarrow x^2 - x - \alpha = 0$

$x_{1,2} = 1 \pm \sqrt{1+4\alpha}$, deci $\lim\limits_{n\to\infty} x_n = 1 + \sqrt{1 + 4\alpha}$

25. $$x_{n+1} - x_n = \frac{1}{2}\left(x_n + \frac{1}{x_n}\right) - x_n = \frac{1}{2}\left(x_n + \frac{1}{x_n} - 2x_n\right) = \frac{1}{2}\left(\frac{1}{x_n} - x_n\right) < 0$$

$x_{n+1} = \frac{1}{2}\left(x_n + \frac{1}{x_n}\right)$ >1(deoarece x_n >0)) $\Rightarrow$ x_n este monoton descrescator.

$1 \le x_n$, x_n este marginit inferior $\Rightarrow$ x_n este convergent. Fie $\lim\limits_{n\to\infty} x_n = x.$ Trecand la limita in egalitatea: $x_{n+1} = \frac{1}{2}\left(x_n + \frac{1}{x_n}\right) \Rightarrow x = \frac{1}{2}\left(x + \frac{1}{x}\right), \Rightarrow x^2 = 1 \Rightarrow x = \pm 1 \Rightarrow \lim\limits_{n\to\infty} x_n = 1.$

26.

$$x_{n+1} - x_n = 1 + \frac{x_n^2}{1+x_n} - x_n = \frac{1}{1+x_n} > 0(\textit{deoarece } x_n > 0)$$

$x_{n+1} > x_n \Rightarrow x_n$ sir monoton crescator. Presupunem ca x_n este marginit superior $\Rightarrow$ x_n este convergent. Fie $\lim\limits_{n\to\infty} x_n = x.$ Trecand la limita in egalitatea:

$x_{n+1} = 1 + \frac{x_n^2}{1+x_n} \Rightarrow x = 1 + \frac{x^2}{1+x} \Rightarrow x + x^2 = 1 + x + x^2 \Rightarrow 1 = 0$ contradictie.

Ceea ce am presupus nu este adevarat $\Rightarrow$ x_n este crescator si nemarginit $\Rightarrow \lim\limits_{n\to\infty} x_n = \infty$

27.

$x_{n+1} - x_n = x_n^2 - 3x_n + 2 = (x_n - 1)(x_n - 2)$

$x_0 = a$

$x_1 = x_0^2 - 2x_0 + 2 = (a-1)^2 + 1$

$x_2 = x_1^2 - 2x_1 + 2 = (x_1 - 1)^2 + 1 = (a-1)^4 + 1 = (a-1)^{2^2} + 1$

....

$x_n = (a-1)^{2^n} + 1$

P(n): $x_n = (a-1)^{2^n} + 1$

P(0): $x_0 = (a-1) + 1 = a$ P(0) adevarata

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P(k): $x_k = (a-1)^{2^k} + 1$

P(k+1): $x_{k+1} = (a-1)^{2^{k+1}} + 1$

$$x_{k+1} = x_k^2 - 2x_k + 2 = (x_k - 1)^2 + 1 = \left[(a-1)^{2^k}\right]^2 + 1 = (a-1)^{2^{k+1}} + 1$$

P(k) $\Rightarrow P(k+1)$ propozitie adevarata $\forall n \in N.$

$$\lim_{n\to\infty} x_k = \lim_{n\to\infty} \left[(a-1)^{2n} + 1\right] = \begin{cases} \infty, \text{ a} > 2 \\ 2, \text{ a} = 2 \\ 1, \text{ } 0 < \text{a} < 2 \\ 2, \text{ a} = 0 \\ \infty, \text{ a} < 0 \end{cases}$$

28.

$$x_1 = \frac{2ax_0}{a + x_0}; \quad x_2 = \frac{2ax_1}{a + x_1} = \frac{2a\,\dfrac{2ax_0}{a + x_0}}{\dfrac{2ax_0}{a + x_0} + a} = \frac{2^2 ax_0}{(1+2)x_0 + a};$$

$$x_3 = \frac{2ax_2}{a + x_2} = \frac{2a\,\dfrac{4ax_0}{3x_0 + a}}{\dfrac{4ax_0}{3x_0 + a} + a} = \frac{8ax_0}{7x_0 + a} = \frac{2^3 ax_0}{(1 + 2 + 2^2)x_0 + a};$$

$$x_n = \frac{2^n ax_0}{(1 + 2 + 2^2 + \cdots + 2^{n-1})x_0 + a}$$

P(n): $x_n = \dfrac{2^n ax_0}{(1 + 2 + 2^2 + \cdots + 2^{n-1})x_0 + a}$

P(1): $x_1 = \dfrac{2ax_0}{x_0 + a}$ adevarata

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P(k): $x_k = \dfrac{2^k ax_0}{(1 + 2 + 2^2 + \cdots + 2^{k-1})x_0 + a}$

P(k+1): $x_{k+1} = \dfrac{2^{k+1} ax_0}{(1 + 2 + 2^2 + \cdots + 2^k)x_0 + a}$

$$x_{k+1} = \frac{2ax_k}{x_k + a} = \frac{2a\,\dfrac{2^k ax_0}{(1 + 2 + 2^2 + \cdots + 2^{k-1})x_0 + a}}{\dfrac{2^k ax_0}{(1 + 2 + 2^2 + \cdots + 2^{k-1})x_0 + a} + a} = \frac{2^{k+1} a^2 x}{\left[(1 + 2 + 2^2 + \cdots + 2^k)x_0 + a\right]a} =$$

$$= \frac{2^{k+1} a\ x}{(1 + 2 + 2^2 + \cdots + 2^k)x_0 + a}$$

$P(k) \Rightarrow P(k+1)$ propozitie adevarata $\forall n \in N$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{ax_0}{\left(\frac{1}{2}+\frac{1}{2^2}+\cdots\frac{1}{2^n}\right)x_0 + a} = \lim_{n\to\infty} \frac{ax_0}{\frac{1}{2}\cdot\frac{\left(\frac{1}{2}\right)^n - 1}{-\frac{1}{2}}x_0 + a} = \frac{ax_0}{a + x_0}$$

29. $$a_{n+2} - a_n = \frac{a_n - a_{n-2}}{\left(\sqrt{1-a_{n+1}} + \sqrt{1-a_{n-1}}\right)\left(\sqrt{1-a_n} + \sqrt{1-a_{n-2}}\right)} \quad 1.$$

$a_{n+1} - a_n = \dfrac{a_{n-1} - a_n}{\sqrt{1-a_n} + \sqrt{1-a_{n-1}}}$, de unde se poate deduce ca, daca $a_{n-1} > a_n$, *atunci* $a_{n+1} > a_n$, iar pentru $a_{n-1} < a_n$, rezulta $a_{n+1} < a_n$

Rezulta ca daca $a_n > a_{n-2}$, atunci $a_{n+2} > a_n$ si daca $a_n < a_{n-2}$, atunci $a_{n+2} < a_n$.

Fie $a_2 - a_1 = \dfrac{1-a-a^2}{\sqrt{1-a}+a}$ si trinomul $f(x) = -x^2 - x + 1$ cu radacinile: $x_1 = \dfrac{\sqrt{5}-1}{2} \in (0,1)$ si

$x_2 = \dfrac{-\sqrt{5}-1}{2} < 0.$

I. Daca $a \in \left(o, \dfrac{\sqrt{5}-1}{2}\right)$, atunci $1-a-a^2 > 0$ deci $a_2 > a_1$ si $a_2 = \sqrt{1-a} > \sqrt{1-\dfrac{\sqrt{5}-1}{2}} = \dfrac{\sqrt{5}-1}{2}$

De asemenea, se verifica prin ridicari la patrat, ca $a_3 < \dfrac{\sqrt{5}-1}{2}$, daca $a < \dfrac{\sqrt{5}-1}{2}$.

Deci presupunem apoi $a_{2k+1} > a_{2k-1}$, conform relatiei 1 va rezulta $a_{2k+3} > a_{2k+1}$ si deci $(a_{2n+1})_{n\geq 1}$ este crescator.

Presupunem $a_{2k+1} < \dfrac{\sqrt{5}-1}{2}$, rezulta ca si $a_{2k+3} < \dfrac{\sqrt{5}-1}{2}$, adica $(a_{2n+1})_{n\geq 1}$ este si marginit, deci convergent, limita calculandu-se folosind relatia de recurenta: $a_{2n+1} = \sqrt{1-\sqrt{1-a_{2n-1}}}$ si rezulta $\lim_{n\to\infty} a_{2n+1} = \dfrac{\sqrt{5}-1}{2}$.

Analog se demonstreaza ca $(a_{2n+1})_{n\geq 1}$ este un sir descrescator si marginit inferior de $\dfrac{\sqrt{5}-1}{2}$.

Deci, este convergent si limita este tot $\dfrac{\sqrt{5}-1}{2}$.

II. Se procedeaza asemanator pentru $a \in \left(\dfrac{\sqrt{5}-1}{2}, 1\right)$.

III. Daca $a = \dfrac{\sqrt{5}-1}{2}$, atunci sirul este constant, adica $a_n = \dfrac{\sqrt{5}-1}{2}$, $\forall n \in N$.

30.

$x_1 = \sqrt{a};\ x_2 = \sqrt{a+\sqrt{a}} \Rightarrow x_2 > x_1.$

Presupunem

$x_n > x_{n-1,}$ atunci $x^2_{n+1} - x^2_n = x^2_n - x^2_{n-1} > 0\ \forall n \in N$, deci sirul $(x_n)_{n\ge1}$ este crescator.

Avem $x_{n+1} = \sqrt{a + x_n} > x_n$ (1),

adica $x^2_n - x_n - a < 0$, *de unde rezulta ca* x_n apartine intervalului determinat de radacinile ecuatiei $x^2 - x - a = 0$

$$x_1 = \frac{1+\sqrt{1+4a}}{2} > 0,\ x_2 = \frac{1-\sqrt{1+4a}}{2} < 0.$$

Deci, $x_n \in \left(0, \frac{1+\sqrt{1+4a}}{2}\right)$. Fiind crescator si marginit, sirul $(x_n)_{n\ge1}$ este convergent.

Trecand la limita in relatia (1), obtinem $x^2 - x - a = 0$ unde $x = \lim\limits_{n\to\infty} x_n$

$$x_1 = \frac{1+\sqrt{1+4a}}{2},\ x_2 = \frac{1-\sqrt{1+4a}}{2}.$$

x_2 nu poate fi limita sirului $(x_n)_{n\ge1}$ int *rucit am ajunge la o contradictie* : limita unui sir crescator mai mica decat toti termenii sirului.

Deci, sirul $(x_n)_{n\ge1}$ are limita egala cu $\frac{1+\sqrt{1+4a}}{2}$

31. Se calculeaza primii termeni ai sirului si se obtine:

$$x_1 = 1, x_2 = \frac{7}{4}, x_3 = \frac{13}{6},\ x_4 = \frac{113}{48}, x_5 = \frac{193}{8}, x_6 = \frac{2311}{960}.$$

Se observa ca $x_1 < x_2 < x_3 < x_4 < x_5$ dar $x_5 > x_6$ si $x_{n+1} - x_n = 1 + \frac{2+n}{2(n+1)}\cdot x_n - x_n = 1 - \frac{n}{2(n+1)}\cdot x_n$

deci, $x_{n+1} < x_n \Leftrightarrow 1 - \frac{n}{2(n+1)}\cdot x_n < 0$, adica $x_n > 2 + \frac{2}{n}$.

Conditia aceasta este indeplinit a pentru $n = 5, x_5 = 2 + \frac{33}{80}$.

Pr*e*sup*unem ca* $x_n > 2 + \frac{2}{n} = \frac{2(n+1)}{n}$ si deci $x_{n+1} = 1 + \frac{2+n}{2(n+1)}\cdot x_n > 2 + \frac{2}{n} > 2 + \frac{2}{n+1}$ intrucat $\frac{2+n}{2(n+1)}\cdot x_n > 1 + \frac{2}{n}$.

Conform principiul ui inductiei complete rezulta ca, $\forall n \in N, n \ge 5, x_n > 2 + \frac{2}{n}$, si deci $x_{n+1} < x_n$, adica sirul este descrescat or. Rezulta ca fiind monoton si marginit, sirul este convergent . Trecand la limita in relatia de recurenta ce defineste sirul, se obtine $\lim\limits_{n\to\infty} x_n = 2$.

32.

a). din $a_1 > 0 \Rightarrow a_2 > 0$ si deci $a_n > 0\ \forall n \ge 1$. Avem inegalitatea : $1 + na^2_n \ge 2\sqrt{n}\cdot a_n$ si deci

$$\frac{1}{1+na^2_n} \le \frac{1}{2\sqrt{n}\cdot a_n}, \text{ prin urmare: } 0 \le \lim_{n\to\infty} a_{n+1} \le \lim_{n\to\infty}\frac{1}{2\sqrt{n}} = 0 \Rightarrow \lim_{n\to\infty} a_n = 0.$$

b).Din $a_{n+1}=\dfrac{a_n}{1+\mathrm{na}_n^2}=\dfrac{1}{\dfrac{1}{a_n}+\mathrm{na}_n}$ *avem* $\mathrm{na}_n=\dfrac{1}{a_{n+1}}-\dfrac{1}{a_n}$ deci, $\lim\limits_{n\to\infty}\mathrm{na}_n=\lim\limits_{n\to\infty}\left(\dfrac{1}{a_{n+1}}-\dfrac{1}{a_n}\right)$ dar

$\lim\limits_{n\to\infty}\mathrm{na}_n=\lim\limits_{n\to\infty}\dfrac{n}{\dfrac{1}{a_n}}=\lim\limits_{n\to\infty}\dfrac{1}{\dfrac{1}{a_{n+1}}-\dfrac{1}{a_n}}.$

Daca sirul cu termenul general $y_n=\mathrm{na}_n>0$ este convergent, atunci : $l=\lim\limits_{n\to\infty}y_n=\lim\limits_{n\to\infty}\dfrac{1}{y_n}=\dfrac{1}{l}\Rightarrow l=1.$

Deci $\lim\limits_{n\to\infty}\mathrm{na}_n=1.$

Convergenta sirului $y_n=\mathrm{na}_n.$

Vom studia mai int *ai monotonia acestui sir.*

$y_{n+1}-y_n=(n+1)a_{n+1}-\mathrm{na}_n=\dfrac{(n+1)a_n}{1+\mathrm{na}_n^2}-\mathrm{na}_n=\dfrac{a_n}{1+\mathrm{na}_n^2}(1-n^2\mathrm{a}_n^2)=\dfrac{a_n}{1+\mathrm{na}_n^2}(1-y_n^2)$ deci,

$y_{n+1}-y_n=a_{n+1}(1-y_n^2).$

Deosebim urmatoarele doua cazuri :

a). $y_n>1\Rightarrow 1-y_n^2<0$ si deci $y_{n+1}-y_n<0$ ceeace inseamna ca sirul y_n este monoton descrescator.

Sa vedem daca $y_{n+1}>1.$

$y_{n+1}-1=\dfrac{(n+1)a_n}{1+\mathrm{na}_n^2}-1=\dfrac{na_n-1-a_n(na_n-1)}{1+\mathrm{na}_n^2}=\dfrac{(y_n-1)(1-a_n)}{1+\mathrm{na}_n^2}>0$ deoarece $y_n>1$ si $0<a_n<\dfrac{1}{2\sqrt{n}}<1.$

deci $y_{n+1}>1.$ Asadar sirul este monoton crescator si marginit, deci convergent $\Rightarrow\lim\limits_{n\to\infty}y_n=1.$

$b).$ $0<y_n<1$ avem $y_{n+1}-y_n=a_{n+1}(1-y_n^2)>0$ si deci $y_{n+1}>y_n$ iar sirul este monoton crescator. Sa aratam ca $y_{n+1}<1;$ $y_{n+1}-1=\dfrac{(y_n-1)(1-a_n)}{1+\mathrm{na}_n^2}<0,$ *deci* $y_{n+1}<1.$

Re *zulta ca sirul* y_n este monoton crescator si marginit, deci convergent.

33. a). Sirul se mai poate scrie:

$\dfrac{x_{n+1}}{2}=arctg\dfrac{2}{\sin x_n}$ *sau* $\mathrm{tg}\dfrac{x_{n+1}}{2}=\dfrac{2}{\sin x_n}$, deoarece $x_1\in(0,\pi)\Rightarrow\sin x_1>0$ si $arctg\dfrac{2}{\sin x_1}\in(0,\dfrac{\pi}{2})\Rightarrow$

$x_2\in(0,\pi).$ Presupunem ca $x_n\in(0,\pi)\Rightarrow\dfrac{2}{\sin x_n}>0$ si deci $arctg\dfrac{2}{\sin x_n}\in(0,\dfrac{\pi}{2}),$ *iar* $x_{n+1}\in(0,\pi).$

Avem, $\sin x_n=\dfrac{2tg\dfrac{x_n}{2}}{1-tg\dfrac{x_n}{2}}$, $\mathrm{a}_n=tg\dfrac{x_n}{2}$, obtinem in acest fel sirul dat de relatia de recurenta :

$$\begin{cases}\mathrm{a}_1>0\\ a_{n+1}=\mathrm{a}_n+\dfrac{1}{\mathrm{a}_n}, x_n=2arctg\,\mathrm{a}_n.\end{cases}$$

Monotonia sirului; $\frac{x_{n+1}}{2}-\frac{x_n}{2}=arctg\ \mathrm{a}_{n+1}-arctg\ \mathrm{a}_n=arctg\frac{\mathrm{a}_{n+1}-\mathrm{a}_n}{1+\mathrm{a}_{n+1}\cdot\mathrm{a}_n}>0$ deoarece $\mathrm{a}_{n+1}-\mathrm{a}_n=\frac{1}{\mathrm{a}_n}>0.$

Asadar, *sirul* x_n este monoton crescator si fiind marginit, rezulta ca x_n este convergent. Ridicam la patrat relatia:

$a_{n+1}=\mathrm{a}_n+\frac{1}{\mathrm{a}_n}$ avem :

$\mathrm{a}_2^2=\mathrm{a}_1^2+2+\frac{1}{\mathrm{a}_1^2}$

$\mathrm{a}_3^2=\mathrm{a}_2^2+2+\frac{1}{\mathrm{a}_2^2}$

....

$\mathrm{a}_n^2=\mathrm{a}_{n-1}^2+2+\frac{1}{\mathrm{a}_{n-1}^2}$

Deci $\mathrm{a}_n^2=\mathrm{a}_1^2+\frac{1}{\mathrm{a}_1^2}+2n-2+\left(\frac{1}{\mathrm{a}_2^2}+\cdots+\frac{1}{\mathrm{a}_{n-1}^2}\right)$: atunci $\mathrm{a}_n^2>2n\ \forall \mathrm{n}\geq 2.$ Deci $\mathrm{a}_n>\sqrt{2n}\Rightarrow$

$\lim_{n\to\infty}\mathrm{a}_n\geq\lim_{n\to\infty}\sqrt{2n}=\infty.$ deci, $\lim_{n\to\infty}\mathrm{x}_n=2\lim_{n\to\infty}arctg\ \mathrm{a}_n=2\frac{\pi}{2}=\pi\Rightarrow\lim_{n\to\infty}\mathrm{x}_n=\pi.$

b). Din

$tg\frac{x_{n+1}}{2}=\frac{2}{\sin x_n}\Rightarrow\sin x_n=\frac{2}{a_{n+1}}$ si $\sqrt{\mathrm{n}}\sin x_n=2\frac{\sqrt{\mathrm{n}}}{a_{n+1}}$ deci, avem de studiat convergenta sirului $\mathrm{y}_n=\frac{a_n}{\sqrt{n}}>0.$

Notam $\mathrm{z}_n=y_n^2=\frac{a_n^2}{n}$

Monotonia sirului z_n :

$$\mathrm{z}_{n+1}-\mathrm{z}_n=\frac{a_{n+1}^2}{n+1}-\frac{a_n^2}{n}=\frac{na_{n+1}^2-(n+1)a_n^2}{n(n+1)}=\frac{n\left(a_{n+1}^2-a_n^2\right)-a_n^2}{n(n+1)}.$$

Dar $a_{n+1}^2-a_n^2=2+\frac{1}{a_n^2}\Rightarrow\mathrm{z}_{n+1}-\mathrm{z}_n=\frac{n\left(2+\frac{1}{a_n^2}\right)-a_n^2}{n(n+1)}$ sau $\mathrm{z}_{n+1}-\mathrm{z}_n=\frac{-a_n^4+2na_n^2+n}{n(n+1)a_n^2}.$

Consideram functia: $f(t)=-t^2+2nt+n$ *care se* anuleaza pentru $\mathrm{t}=\mathrm{n}\pm\sqrt{\mathrm{n}^2-n}$ *unde* $t=a_n^2>2n.$

dar $t>2n>\mathrm{n}+\sqrt{\mathrm{n}^2-n}$ si deci $f(t)<0.$

Re*zulta ca* $\mathrm{z}_{n+1}-\mathrm{z}_n<0$ iar sirul z_n este monoton descrescator si fiind sir cu termeni pozitivi, este convergent.

Deci, $\lim_{n\to\infty}\mathrm{z}_n=l$ exista si este finita.

Deci $\mathrm{l}=\lim_{n\to\infty}\mathrm{z}_n=\lim_{n\to\infty}\frac{a_n^2}{n}=\lim_{n\to\infty}\frac{a_{n+1}^2-a_n^2}{n+1-n}=\lim_{n\to\infty}(a_{n+1}^2-a_n^2)=\lim_{n\to\infty}\left(2+\frac{1}{a_n^2}\right)=2.$

Deci $\lim_{n\to\infty}\mathrm{y}_n=\lim_{n\to\infty}\sqrt{\mathrm{z}_n}=\sqrt{2}\Rightarrow\lim_{n\to\infty}\sqrt{\mathrm{n}}\sin x_n=\sqrt{2}.$

34.

$f_n(x)=f\left(f_{n-1}(x)\right)=\frac{2f_{n-1}(x)}{1+f_{n-1}^2(x)}$; notam $f_n(x)=a_n$. Deci: $a_n=\frac{2a_{n-1}}{1+a_{n-1}^2}$, $\mathrm{a}_1=\frac{2x}{1+x^2}$, $\mathrm{n}\in\mathrm{N}^*.$

1). $|a_n|=\frac{2|a_{n-1}|}{1+|a_{n-1}|^2}\leq 1$ deci a_n sir marginit.

2). $a_n - a_{n-1} = \dfrac{2a_{n-1}}{1+a_{n-1}^2} - a_{n-1} = a_{n-1} \cdot \dfrac{1-a_{n-1}^2}{1+a_{n-1}^2}$.

a) Daca $x < 0, n \in N^*$ se arata prin inductie ca $a_n < 0$.

Deci $: a_{n-1} < 0, 1-a_{n-1}^2 > 0, 1+a_{n-1}^2 > 0 \Rightarrow a_n - a_{n-1} < 0 \Rightarrow a_n < a_{n-1} \Rightarrow a_n - \textit{sir monoton descrescator} \Rightarrow$
$a_n - \textit{sir}$ monoton si marginit $\Rightarrow a_n - \textit{sir}$ convergent.

b). $x = 0, a_n = 0$.

c). $x > 0$ $n \in N^*$ se arata prin inductie ca $a_n > 0$, deci $a_n - a_{n-1} > 0 \Rightarrow a_n > a_{n-1} \Rightarrow a_n$ sir monoton crescator;

a_n sir monoton si marginit $\Rightarrow a_n$ convergent. Fie $\lim\limits_{n\to\infty} a_n = a \Rightarrow a = \dfrac{2a}{1+a^2} \Rightarrow a^3 + a - 2a = 0 \Leftrightarrow$

$$a^3 - a = 0 \Leftrightarrow a = 0, a = 1, a = -1 \Rightarrow \lim_{n\to\infty} a_n = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0. \end{cases} \text{ Deci } \lim_{n\to\infty} f_n(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0. \end{cases}$$

35.

$$a_2 = \frac{b_1}{a_1}, \qquad b_2 = \frac{b_1 - 1}{a_1 - 1},$$
$$a_3 = \frac{a_1}{b_1}\left(\frac{b_1 - 1}{a_1 - 1}\right), \qquad b_3 = \frac{a_1}{a_1 - 1},$$
$$a_4 = \frac{b_1}{a_1 - 1}, \qquad b_4 = \frac{b_1}{b_1 - a_1},$$
$$a_5 = \frac{b_1 - 1}{b_1 - a_1}, \qquad b_5 = \frac{a_1(b_1 - 1)}{b_1 - a_1},$$
$$a_6 = a_1, \qquad b_6 = b_1.$$

Sirurile a_n, b_n, sunt periodice si perioada este egala cu 5.

Pentru ca cele doua siruri sa fie convergente este necesar
si suficient ca cele doua siruri sa fie constante, adica $: a_1 = a_2 = a_3 = a_4 = a_5, b_1 = b_2 = b_3 = b_4 = b_5$.

$\Rightarrow a_1^2 - a_1 - 1 = 0 \Rightarrow (a_1)_1 = \dfrac{1+\sqrt{5}}{2} > 0$, si $b_1 = \dfrac{3+\sqrt{5}}{2}$,

iar $(a_1)_2 = \dfrac{1-\sqrt{5}}{2} < 0$. *Valorile cautate pentru* a_1, b_1

sant $a_1 = \dfrac{1+\sqrt{5}}{2}, \quad b_1 = \dfrac{3+\sqrt{5}}{2}$.

Se verifica apoi ca, luand pentru a_1, b_1, valorile gasite, toti termenii vor fi egali.

36.

Relatia prin care este definit sirul , se mai poate scrie si astfel: $a_{n+1} = \dfrac{a_n - 1}{a_n + 1}$

$$a_1 = \frac{a_0 - 1}{a_0 + 1}; a_2 = \frac{a_1 - 1}{a_1 + 1} = -\frac{1}{a_0}; a_3 = \frac{a_2 - 1}{a_2 + 1} = -\frac{1}{a_1}, a_4 = \frac{a_3 - 1}{a_3 + 1} = a_o$$

Presupunem ca: $a_n = a_{n-4}, a_{n-1} = a_{n-5}, a_{n-2} = a_{n-6}, a_{n-3} = a_{n-7}$.

Trebuie aratat ca: $a_{n+1} = a_{n-3}, a_{n+2} = a_{n-2}, a_{n+3} = a_{n-1}, a_{n+4} = a_n$.

$a_{n+1} = \frac{a_n - 1}{a_n + 1} = \frac{a_{n-4} - 1}{a_{n-4} + 1} = a_{n-3}$, $a_{n+2} = \frac{a_{n+1} - 1}{a_{n+1} + 1} = \frac{a_{n-3} - 1}{a_{n-3} + 1} = a_{n-2}$

$a_{n+3} = \frac{a_{n+2} - 1}{a_{n+2} + 1} = \frac{a_{n-2} - 1}{a_{n-2} + 1} = a_{n-1}$, $a_{n+4} = \frac{a_{n+3} - 1}{a_{n+3} + 1} = \frac{a_{n-1} - 1}{a_{n-1} + 1} = a_n$;

deci sirul a_n este periodic cu perioada egala cu 4.

Fiind un sir periodic, a_n *nu* este un sir convergent

Daca $a_{1976} = 40$ si $1976 = 4k \Rightarrow a_{1976} = a_0 = .40.$

Sirul a_n *va avea* urmatoarea configuratie: $a_0 = 40, a_1 = \frac{39}{41},$

$a_2 = -\frac{1}{40}, a_3 = -\frac{41}{39}, a_4 = a_0 = 40.$

37.

Fie x_n un sir periodic de perioada p, deci $x_{kp+r} = x_r, \forall k \in N$ si $r = \overline{0, p-1}$.

Fie atunci subsirurile:

1). $x_0, x_p, x_{2p}, \cdots x_{np}, \to x_0$

2). $x_1, x_{p+1}, x_{2p+1}, \cdots x_{np+1}, \to x_1$

....

p). $x_{p-1}, x_{2p-1}, \cdots x_{np+p-1}, \to x_{p-1}$.

Daca sirul este periodic de perioada p, cel putin doua d int*re numerele* $x_0, x_1 \cdots x_{p-1}$ sunt diferite , deci x_n contine doua subsiruri convergente spre limite diferite, deci nu poate fi un sir convergent.

38.

Fie $k \in N$, impar $\Rightarrow a_n = a_{n-1} - a_{n-2} + a_{n-3} - \cdots + a_{n-k}$

$$a_{n+1} = a_n - a_{n-1} + a_{n-2} - \cdots + a_{n-k+1}$$

....

$$a_{n+1} = a_{n-k}, \forall n \in N \text{ si } k \in N, \text{dat.}$$

Fie $k \in N$, par $\Rightarrow$ $a_n = a_{n-1} - a_{n-2} + a_{n-3} - \cdots - a_{n-k}$

$$a_{n+1} = a_n - a_{n-1} + a_{n-2} - \cdots - a_{n-k+1}$$

....

$$a_{n+1} = -a_{n-k}, \forall n \in N \text{ si } k \in N, \text{dat.}$$

Pentru $n = k + 1$ $\quad a_{k+2} = -a_1$

....

$a_{2k+3} = -a_{k+2} = a_1$ deci sirul a_n este periodic cu perioada $(2k+2)$.

b). Pentru k = 2 $\Rightarrow a_n = a_{n-1} - a_{n-2}$, adica a_n este un sir Fibonaci. Ecuatia caracteristica : $x^2 - x + 1 = 0 \Rightarrow$

$$x_1 = \frac{1+i\sqrt{3}}{2}, \quad x_2 = \frac{1-i\sqrt{3}}{2}$$

$$a_n = a\cos\frac{n\pi}{3} + b\sin\frac{n\pi}{3}$$

$$n=1\begin{cases} a_1 = a\cos\frac{\pi}{3} + b\sin\frac{\pi}{3} \\ a_2 = a\cos\frac{2\pi}{3} + b\sin\frac{2\pi}{3} \end{cases} \Rightarrow \begin{cases} a = a_1 - a_2 \\ b = \frac{a_1 + a_2}{\sqrt{3}} \end{cases}$$

$$\Rightarrow a_n = (a_1 - a_2)\cos\frac{n\pi}{3} + \frac{a_1 + a_2}{\sqrt{3}}\sin\frac{n\pi}{3} = \frac{2\sqrt{3}}{3}\left[a_1\cos\frac{(2n-1)\pi}{6} - a_2\cos\frac{(2n+1)\pi}{6}\right]$$

39.

$$I_n = \int_n^{n+1}\frac{2x-1}{x}dx\int_n^{n+1}\left(2-\frac{1}{x}\right)dx = [2x - \ln x]_n^{n+1} = 2n+2-\ln(n+1)-2n+\ln n = 2-\ln\left(1+\frac{1}{n}\right).$$

$$a).\ I_{n+1} - I_n = 2 - \ln\frac{n+2}{n+1} - 2 + \ln\frac{n+1}{n} = \ln\left(1+\frac{1}{n(n+1)}\right) > 0 \Rightarrow I_{n+1} > I_n \Rightarrow I_n\textit{sir monoton crescator}.$$

$$b).\ 2 - \ln 2 < I_n \le 2 \text{ deoarece } \ln\left(1+\frac{1}{n}\right). > 0.$$

$$c).\ \lim_{n\to\infty} n(2 - I_n) = \lim_{n\to\infty} n\ln\left(1+\frac{1}{n}\right) = \lim_{n\to\infty}\ln\left(1+\frac{1}{n}\right)^n = \ln e = 1.$$

40.

$$a).\ v_n = \sum_{k=1}^{n}\frac{1}{(2k-1)(2k+1)} = \frac{n}{2n+1}$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\left(1+\frac{1}{2n+1}\right)^{\frac{n(2n+1)}{2n+1}} = e^{\lim\limits_{n\to\infty}\frac{n}{2n+1}} = e^{\frac{1}{2}}.$$

$$b).\ \lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{2n+4}{2n+3}\left[\frac{2(n+2)}{2n+3}\cdot\frac{2n+1}{2(n+1)}\right]^n = \lim_{n\to\infty}\frac{2n+4}{2n+3}\left(1-\frac{1}{2n^2+5n+3}\right)^n =$$

$$= \lim_{n\to\infty}\frac{2n+4}{2n+3}\lim_{n\to\infty}\left(1-\frac{1}{2n^2+5n+3}\right)^{\frac{2n^2+5n+3}{-1}\cdot\frac{-n}{2n^2+5n+3}} = 1\cdot e^0 = 1$$

41.

$$b_k = \lim_{n\to\infty}\left(\frac{n^2 - nk + 1}{n^2+1}\right)^{\frac{n^2+n+1}{n+1}} = e^{-k}$$

$$c_n = \sum_{k=1}^{n} e^{-k} = \sum_{k=1}^{n}\frac{1}{e^k} = \frac{1}{e}\cdot\frac{1-\frac{1}{e^n}}{1-\frac{1}{e}} = \frac{e^n - 1}{(e-1)e^n}$$

$$\lim_{n\to\infty} c_n = \frac{1}{e-1}\cdot\lim_{n\to\infty}\frac{e^n-1}{e^n} = \frac{1}{e-1}$$

42.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2+6n+12}{(n+1)(n+2)} \cdot \frac{1}{n \cdot 2^{n+1}} = 0.$$

$$b_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} \left[\frac{k+3}{k(k+1)\cdot 2^k} - \frac{k+4}{(k+1)(k+2)\cdot 2^{k+1}} \right] = 1 - \frac{n+4}{(n+1)(n+2)\cdot 2^{n+1}} \Rightarrow \lim_{n\to\infty} b_n = 1$$

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} b_n^{2^n} = \lim_{n\to\infty} \left[1 - \frac{n+4}{(n+1)(n+2)\cdot 2^{n+1}} \right]^{2^n} e^{\lim\limits_{n\to\infty} \frac{-(n+4)}{2(n+1)(n+2)}} = e^0 = 1$$

43.

$$a).\ \mathrm{a}_2 = \frac{a}{\sqrt{a^2+1}};\ \mathrm{a}_3 = \frac{a}{\sqrt{1+2a^2}};\ \mathrm{a}_4 = \frac{a}{\sqrt{1+3a^2}}.$$

$$\text{Presupunem ca } \mathrm{a}_{n-1} = \frac{a}{\sqrt{1+(n-2)a^2}}. \Rightarrow \mathrm{a}_n = \frac{a}{\sqrt{1+(n-1)a^2}}$$

$$\lim_{n\to\infty} a_n = a \lim_{n\to\infty} \frac{1}{\sqrt{1+(n-1)a^2}} = 0.$$

$$b).\ \mathrm{b}_n = \left(na_{n+1}^2\right)^{-\alpha n-\beta} = \left(1+\frac{1}{n\cdot a^2}\right)^{\alpha n+\beta};\ \lim_{n\to\infty} b_n = \lim_{n\to\infty}\left(1+\frac{1}{n\cdot a^2}\right)^{\alpha n+\beta} = e^{\lim\limits_{n\to\infty}\frac{\alpha n+\beta}{na^2}} = e^{\frac{\alpha}{a^2}}$$

44.

$$a_n = \frac{3^{3n}}{\left(3+\frac{1}{n}\right)^{3n}} = \frac{3^{3n}}{3^{3n}\left(1+\frac{1}{3n}\right)^{3n}} = \frac{1}{\left(1+\frac{1}{3n}\right)^{3n}}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{3n}\right)^{3n}} = \frac{1}{e}.$$

45.

$$a_1 = \frac{1}{2\cdot 1!};\ a_2 = \frac{2^2\cdot 2!-1}{2^2\cdot 2!};\ a_3 = \frac{2^3\cdot 3!-1}{2^3\cdot 3!}.\ \ \mathrm{P(n)} : \mathrm{a}_n = \frac{2^n\cdot n!-1}{2^n\cdot n}.$$

$$P(1) : a_1 = \frac{1}{2\cdot 1!};....;P(k) : \mathrm{a}_k = \frac{2^k\cdot k!-1}{2^{kn}\cdot k!};P(k+1) : \mathrm{a}_{k+1} = \frac{2^{k+1}(k+1)!-1}{2^{k+1}(k+1)!}$$

$$\mathrm{a}_n = \frac{2^n\cdot n!-1}{2^n\cdot n}$$

$$Deci\ \mathrm{a}_{k+1} = a_k + \frac{2k+1}{2^{k+1}(k+1)!} = \frac{2^k\cdot k!-1}{2^{kn}\cdot k!} + \frac{2k+1}{2^{k+1}(k+1)!} = \frac{2^{k+1}(k+1)!-1}{2^{kn+1}(k+1)!}.$$

$$P(k)\to P(k+1) \Rightarrow P(n) \text{ adevarata } \forall \mathrm{n} \in \mathrm{N}^*. \text{Rezulta ca } \mathrm{a}_n = \frac{2^n\cdot n!-1}{2^n\cdot n}$$

46. *Notam* $\sqrt[n]{a_n} = b_n$. Atunci relatia de definitie a sirului: $\frac{1}{\mathrm{b}_{n+1}-1} - \frac{1}{\mathrm{b}_n-1} = 1, \mathrm{b}_1 = 2 \Rightarrow b_{n+1} = \frac{2b_n-1}{b_n}$:

$$b_1 = 2, \mathrm{b}_2 = \frac{3}{2}, \mathrm{b}_3 = \frac{4}{3}. \text{Presupunem ca} : \mathrm{b}_n = \frac{n+1}{n}, \text{atunci } \mathrm{b}_{n+1} = \frac{2b_n-1}{b_n} = \frac{2\frac{n+1}{n}-1}{\frac{n+1}{n}} = \frac{n+2}{n+1}.$$

Conform principiului inductiei complete, forma propusa pentru termenul general al sirului b_n este adevarata.

$$\mathrm{a}_n = (b_n)^n = \left(1+\frac{1}{n}\right)^n. \text{Deci, sirul } \mathrm{a}_n \textit{ este} \text{ sirul numarului "e".}$$

47.

$$Calculam : \left|x_{n+p} - x_n\right| = \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(n+p)^2} < \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} + \cdots +$$

$$+ \frac{1}{(n+p-1)(n+p)} = \frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+1} - \frac{1}{n+2} + \cdots + \frac{1}{n+p-1} - \frac{1}{n+p} = \frac{1}{n} - \frac{1}{n+p} = \frac{p}{n(n+p)} < \frac{1}{n};$$

deci, $\forall \varepsilon > 0\ \exists \mathrm{n}_{\varepsilon} = \left[\frac{1}{\varepsilon}\right]$ astfel incat $\forall \mathrm{n} > \mathrm{n}_{\varepsilon}$ si $\forall \mathrm{p} \in \mathrm{N}$ atunci $\left|x_{n+p} - x_n\right| < \varepsilon.$

Fiind satisfacut criteriul lui Cauchy, sirul este convergent.

48.

A arata ca x_n este un sir convergent este echivalent cu a arata ca sirul este fundamental.

x_n este fundamental $\Leftrightarrow \forall \varepsilon > 0, \exists \mathrm{n}_0 \in N$ cu proprietatea ca

$$\left|x_{n+p} - x_n\right| = \left|\frac{\cos a_{n+1}}{(n+1)(n+2)} + \frac{\cos a_{n+2}}{(n+2)(n+3)} + \cdots + \frac{\cos a_{n+p}}{(n+p)(n+p+1)}\right| < \left|\frac{\cos a_{n+1}}{(n+1)(n+2)}\right| +$$

$$+ \left|\frac{\cos a_{n+2}}{(n+2)(n+3)}\right| + \cdots + \left|\frac{\cos a_{n+p}}{(n+p)(n+p+1)}\right| < \frac{1}{(n+1)(n+2)} + \frac{1}{(n+2)(n+3)} + \cdots +$$

$$+ \frac{1}{(n+p)(n+p+1)} = \frac{1}{n+1} - \frac{1}{n+p+1} < \frac{1}{n+1} < \varepsilon$$

$\left(y_n = \frac{1}{n+1} \to 0\right)$; fie $\varepsilon > \mathrm{o}$ si $\mathrm{n}_0 = \left[\frac{1}{\varepsilon}\right]$, deci $\mathrm{n}_0 + 1 > \frac{1}{\varepsilon}$;

atunci, $\forall\, \mathrm{n} > \mathrm{n}_0$ si $\mathrm{p} \in \mathrm{N}$, $\left|x_{n+p} - x_n\right| < \frac{1}{n+1} < \frac{1}{n_0 + 1} < \varepsilon$; *deci*, fiind sir fundamental, este convergent

49.

$$\left|x_{n+p} - x_n\right| = \left|\frac{\cos(n+1)!}{(n+1)(n+2)} + \frac{\cos(n+2)!}{(n+2)(n+3)} + \cdots + \frac{\cos(n+p)!}{(n+p)(n+p+1)}\right| \le \sum_{k=1}^{p} \frac{\left|\cos(n+k)!\right|}{(n+k)(n+k+1)}$$

$$\le \sum_{k=1}^{p} \frac{1}{(n+k)(n+k+1)} = \frac{1}{n+1} - \frac{1}{n+p+1} < \frac{1}{n+1} < \varepsilon$$

$y_n = \frac{1}{n+1} \to 0 \Rightarrow \forall \varepsilon > 0, \exists\, \mathrm{n}_0 = \left[\frac{1}{\varepsilon}\right]$, a.i. $\forall \mathrm{n} > \mathrm{n}_0, \left|\mathrm{y}_{\mathrm{n}}\right| < \varepsilon \Rightarrow \frac{1}{n+1} < \varepsilon;$

deci, fiind un sir fundamental este convergent.

50.

$$\mathrm{a}_n = \frac{3}{n} \sum_{k=1}^{n} \sqrt{\frac{n}{n+3(k-1)}} =_n = \frac{3}{n} \sum_{k=1}^{n} \frac{1}{\sqrt{1 + \frac{3(k-1)}{n}}}.$$

Fie $\mathrm{f(x)} = \frac{1}{\sqrt{1+x}}$,

int*ervalul* $[0,3]$ cu diviziunea $\mathrm{x}_{\mathrm{k}} = \frac{3k}{n}$ $(\mathrm{k} = 0,1,2 \cdots \mathrm{n})$ si sistemul de puncte intermediare

$\xi_{\mathrm{k}} = \frac{3k}{n}$, $(\mathrm{k} = 1,2 \cdots \mathrm{n})$, $(\xi_{\mathrm{k}} = \mathrm{x}_{\mathrm{k-1}})$

$$\int_0^3 \frac{1}{\sqrt{1+x}}dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1}) = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_{k-1})(x_k - x_{k-1}) = \lim_{n\to\infty}\frac{3}{n}\sum_{k=1}^{n}\frac{1}{\sqrt{1+\frac{3(k-1)}{n}}} = \lim_{n\to\infty} a_n.$$

$$Deci \lim_{n\to\infty} a_n = \int_0^3 \frac{1}{\sqrt{1+x}}dx = \left[2\sqrt{1+x}\right]_0^3 = 4-2=2$$

51. Fie

$$f(x)=\sqrt{1+x} \text{ si intervalul} [0,1] \text{cu diviziunea } x_k = \frac{k}{n} (k=0,1,\cdots,n) \text{ si}$$

$$\text{sistemul de puncte intermediare } \xi_k = x_k, (k=1,2,\cdots,n).$$

$$\int_0^1 \sqrt{1+x}dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1}) = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)(x_k - x_{k-1}) = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\sqrt{1+\frac{k}{n}} = \lim_{n\to\infty} a_n.$$

$$Deci \lim_{n\to\infty} a_n = \int_0^1 \sqrt{1+x}dx = \int_1^{\sqrt{2}} 2t^2 dt = 2\left(\frac{2\sqrt{2}-1}{3}\right) = \frac{2\left(2\sqrt{2}-1\right)}{3}.$$

52. Fie functia f(x)=sinx si intervalul $[0,\pi]$ cu diviziunea

$$x_k = \frac{k\pi}{n}, \text{ k} = (0,1,\cdots,\text{n}) \text{ si sistemul de puncte intermediare } \xi_k = x_{k-1}, \text{ k} = (1,\cdots,\text{n}).$$

$$\int_0^\pi \sin x dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1}) = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_{k-1})(x_k - x_{k-1}) = \lim_{n\to\infty}\frac{\pi}{n}\sum_{k=1}^{n}\sin\frac{(k-1)\pi}{n} = \lim_{n\to\infty} a_n.$$

$$Deci \lim_{n\to\infty} a_n = \int_0^\pi \sin x dx = [-\cos x]_0^\pi = 1+1=2.$$

53.

$$Fie \text{ f(x)} = \frac{1}{(1+\text{x})^2} \text{ si intervalul } [0,1] \text{ cu diviziunea } x_k = \frac{k}{n}, \quad \text{k} = (0,1,2,\cdots,\text{n}) \text{ si}$$

$$\text{sistemul de puncte intermediare}: \xi_k = x_k \quad \text{k} = (1,2,\cdots,\text{n}).$$

$$\int_0^1 \frac{1}{(1+\text{x})^2}\text{dx} = \lim_{n\to\infty}\sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1}) = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)(x_k - x_{k-1}) = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{\left(1+\frac{k}{n}\right)^2} = \lim_{n\to\infty} a_n.$$

$$\lim_{n\to\infty} a_n = \int_0^1 \frac{1}{(1+\text{x})^2}dx = \left[-\frac{1}{1+x}\right]_0^1 = -\frac{1}{2}+1 = \frac{1}{2}.$$

54.

$$a). \text{ Fie f(x)} = \text{e}^x \cos^2 x \text{ si } [0,1] \text{ cu diviziunea } x_k = \frac{k}{n} \quad \text{k} = 0,1,2\cdots,\text{n si}$$

$$\text{sistemul de puncte intermediare}: \xi_k = x_k, \text{k} = 1,2\cdots,\text{n}$$

$$\int_0^1 \text{e}^x \cos^2 x dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1}) = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)(x_k - x_{k-1}) = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\text{e}^{\frac{k}{n}}\cos^2\frac{k}{n} = \lim_{n\to\infty} a_n.$$

$$\lim_{n\to\infty} a_n = \int_0^1 \text{e}^x\cos^2 x dx = [\text{e}^x\cos^2 x]_0^1 + \int_0^1 \text{e}^x \sin 2x dx = [\text{e}^x\cos^2 x]_0^1 + [\text{e}^x \sin 2x]_0^1 - \int_0^1 \text{e}^x\cdot 2\cos 2x dx =$$

$$[\text{e}^x\cos^2 x]_0^1 + [\text{e}^x\sin 2x]_0^1 - 2\int_0^1 \text{e}^x(2\cos^2 x - 1)dx = [\text{e}^x\cos^2 x]_0^1 + [\text{e}^x\sin 2x]_0^1 - 4\int_0^1 \text{e}^x\cos^2 x dx + 2\int_0^1 \text{e}^x dx.$$

$$\int_0^1 \text{e}^x\cos^2 x dx = \frac{1}{5}\left\{[\text{e}^x\cos^2 x]_0^1 + [\text{e}^x\sin 2x]_0^1 + 2[e^x]_0^1\right\} = \frac{1}{5}\left(e\cos^2 1 + e\sin 2 + 2e - 3\right).$$

$$\text{Deci} \lim_{n\to\infty} a_n = \frac{1}{5}\left(e\cos^2 1 + e\sin 2 + 2e - 3\right)$$

b)

$$\int_0^1 \mathrm{e}^x \sin^2 x dx = \int_0^1 \mathrm{e}^x (1-\cos^2 x)dx = e-1-\int_0^1 \mathrm{e}^x \cos^2 x dx = e-1-\frac{1}{5}\left(e\cos^2 1+e\sin 2+2e-3\right)=$$

$$-\frac{2}{5}-\frac{e}{5}(\cos^2 1+\sin 2-3).$$

$$\int_0^1 \mathrm{e}^x \sin^2 x dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(\xi_k)(x_k-x_{k-1}) = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)(x_k-x_{k-1}) = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\mathrm{e}^{\frac{k}{n}}\sin^2\frac{k}{n} = \lim_{n\to\infty} b_n.$$

$$\textit{Deci } \lim_{n\to\infty} b_n = -\frac{2}{5}-\frac{e}{5}(\cos^2 1+\sin 2-3).$$

55.

Pentru aflarea acestei limite ne vom folosi de identitatea lui Catalan:

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2n-1}-\frac{1}{2n}=\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}, \text{ care se obtine din identitatea :}$$

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{2n-1}+\frac{1}{2n}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n-1}+\frac{1}{2n};$$

$$\textit{scazand in ambii membri } -2\left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2n}\right)$$

Deci, problema se reduce la calcularea :

$$\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\right). \text{ Prin integrare se obtine un rezultat imediat.}$$

Consideram functia $\mathrm{f(x)} = \frac{1}{1+\mathrm{x}}$ continua pe intervalul $[0,1]$.

Fiind continua pe $[0,1]$, f este integrabila, si deci considerand diviziunile: $0, \frac{1}{\mathrm{n}}, \frac{1}{\mathrm{n}}, \cdots, \frac{n}{n}$ si sistemul de puncte intermediare: $\xi_{\mathrm{k}} = \frac{k}{n}$ $\mathrm{k} = (1,2,\cdots,\mathrm{n})$, vom avea :

$$\int_0^1 \mathrm{f(x)dx} = \lim_{n\to\infty}\sum_{k=1}^{n} f(\xi_k)(x_k-x_{k-1}) = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\cdot\frac{1}{1+\frac{k}{n}} = \lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\right).$$

$$\int_0^1 f(x)dx = \ln[1+x]_0^1 = \ln 2. \text{ Deci } \lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\right) = \ln 2.$$

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